Effective theory of systems coupled strongly to rapidly-varying external sources.

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We consider quantum systems which interact strongly with a rapidly varying environment and derive a Schrödinger-like equation which describes the time evolution of the average wave function. We show that the corresponding Hamiltonian can be taken to be Hermitian provided all states are rotated using an appropriate unitary transformation. The formalism is applied to a variety of systems and is compared and contrasted with related results describing stochastic resonances.

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I. INTRODUCTION

The study of quantum systems which interact strongly with their environment often presents serious challenges due to the possibility that these interactions cannot be neglected and, in addition, also vary rapidly and randomly with time [1]. The effects of such external fields is often unavoidable and interesting and can lead to unexpected phenomena such as, for example, those studied under the blanket term of stochastic resonances [2]. In this paper we will study one subset of such systems.

We will assume that the interactions with the environment are described by a time-dependent contribution to the Hamiltonian denoted by $H'(t)$ which cannot be treated perturbatively. In analogy with a similar problem in mechanics [3] we will assume self-consistently that the states of the system can be decomposed into a sum of slowly varying modes and high-frequency components of small amplitude. Using this decomposition we will show that the time evolution of the slow modes is determined by an effective Hamiltonian which, to leading order, depends quadratically on the external interaction H'. The formalism assumes that the time scales associated with the interactions with the environment are much shorter than all other frequencies in the problem. Denoting by Ω a typical frequency of the interaction H', we will obtain a solution to Schrödinger's equation as a series in $1/\Omega$.

The effective Hamiltonian describes the average time evolution of the system and can exhibit resonances under some special circumstances which will be illustrated using simple examples. It is also worth noting that the same formalism can be applied to *any* system evolving according to a Schrödinger-like equation assuming that the operators corresponding to the Hamiltonian contain terms which vary rapidly in the evolution parameter. We also provide examples of this type of generalization: using geometrical optics we study light-ray propagation in a random media, and, we determine the effects of a time-independent potential which varies rapidly with position on the wave functions of a non-relativistic particle.

The paper is organized as follows: in section II we give a description of the formalism and find the effective Hamiltonian that will be used in the applications. The behavior of the effective Hamiltonian under unitary transformations is studied in section III; the formalism is then applied to various illustrative examples in section IV. In section V we give an alternative view of the problem in terms of the Fokker-Planck equation and the results are then compared and contrasted with the formalism used in deriving the standard stochastic resonances (section VI). Paring comments and conclusions are presented in section VII. Finally, a mathematical detail is relegated to the appendix.

II. QUANTUM SYSTEMS WITH RAPIDLY-VARYING EXTERNAL FIELDS

We consider a generic quantum system with a Hamiltonian of the form

$$H = H_0 + H', \tag{1}$$

where H' is time dependent with characteristic frequencies assumed larger than all the other energy scales in the system (we take units where $\hbar = 1$). In general we will also allow H_0 to vary with time, but with the restriction that the time scale(s) associated with H_0 are much smaller than those associated with the time variation of H'. In addition we assume that H' is larger than H_0 so that, symbolically

$$H_0,\ \dot{H}_0/H_0 < H' < \dot{H}'/H'. \tag{2}$$

More specifically we assume that H' admits a Fourier expansion of the type

$$H' = \sum_{|\omega|>\Omega} H_\omega e^{-i\omega t}, \tag{3}$$

where the sum is over a set of frequencies $\{\omega\}$ such that the differences also obey $|\omega - \omega'| \geq \Omega$. In general we will allow the Fourier coefficients H_ω to be time-dependent, but, as for H_0, we assume that the corresponding frequencies are small compared to Ω. Henceforth "slow" will mean "of frequency $\ll \Omega$".

In solving the Schrödinger equation for such a system we will assume that the wave function can be separated into a slowly varying piece ψ and a rapidly varying (frequency $\gtrsim \Omega$) "ripple" χ of small amplitude,

$$\Psi = \underbrace{\psi}_{\text{slow}} + \underbrace{\chi}_{\text{fast}}. \tag{4}$$

We will then match slow and fast terms, noting that even though $|\chi|$ is small, $\dot{\chi}$ can be large.

We will assume that all quantities can be written as the sum of a slowly varying piece (containing frequencies $\ll \Omega$) and a fast piece (in special cases one or the other may vanish). It then proves convenient to introduce the following notation: for any quantity Ξ

$$\langle \Xi \rangle : \text{slow part of } \Xi. \tag{5}$$

For example, $\langle \Psi \rangle = \psi$, $\langle H \rangle = H_0$, $\langle H' \rangle = 0$.

To solve Schrödinger's equation for this class of systems we consider a typical term in (3) and define the expansion parameter

$$\eta \sim \frac{|H_\omega|}{\Omega} \tag{6}$$

(alternatively $\eta \sim |\int dtH'|$); the previous restrictions imply that $\eta < 1$. We will now assume that the wave function has an expansion in powers of η:

$$\Psi = \psi + \chi_1 + \chi_2 + \cdots, \quad \chi_n \sim \eta^n. \tag{7}$$

This expansion is useful for $n \ll 1/\eta$, beyond this order we typically obtain $n-$fold products of slowly varying quantities which can generate terms with frequency $\sim \Omega$, and the separation between slowly and rapidly varying terms cannot be maintained. These effects are, however, subdominant since their *amplitude* are suppressed by a small factor $\propto \eta^n \ll 1$.

Substituting H from (1) and Ψ from (4) in Schrödinger's equation

$$H\Psi = i\dot{\Psi}, \tag{8}$$

we find, to lowest order in $1/\Omega$

$$i\dot{\psi} + i\dot{\chi}_1 + O(\Omega\eta^2) = H_0\psi + H'\psi + O(H_0\eta, \Omega\eta^2), \tag{9}$$

whence

$$i\dot{\chi}_1 = H'\psi, \qquad i\dot{\psi} = H_0\psi. \tag{10}$$

Since ψ is slowly varying, the first equation can be solved to this order in η by taking ψ constant, namely,

$$\chi_1 = \mathcal{U}\psi; \qquad \mathcal{U} = \left(\frac{1}{i}\int dtH'\right) = \sum_{|\omega|>\Omega} \frac{1}{\omega} H_\omega e^{-i\omega t}. \tag{11}$$

To the next order we write

$$\Psi = \psi + \mathcal{U}\psi + \chi_2 + O(\eta^3), \tag{12}$$

and obtain

$$i\dot{\psi} + i\mathcal{U}\dot{\psi} + i\dot{\chi}_2 = H_0\psi + H_0\mathcal{U}\psi + H'\mathcal{U}\psi + O(\Omega\eta^3, H_0\eta^2). \tag{13}$$

Note that $H'\mathcal{U}$ contains both slow and fast terms. Using the notation (5) we find

$$\begin{aligned} i\dot{\psi} &= \left(H_0 + \langle H'\mathcal{U}\rangle\right)\psi, \\ i\dot{\chi}_2 &= \left([H_0,\mathcal{U}] + H'\mathcal{U} - \langle H'\mathcal{U}\rangle\right)\psi, \end{aligned} \tag{14}$$

where the second equation can be solved (to the present order in η) by neglecting the time variation in H_0 and ψ.

To this order the *average* wave function then obeys a Schrödinger-like equation with an effective Hamiltonian [4]

$$H_{\rm eff} = H_0 + \langle H'\mathcal{U}\rangle\,. \tag{15}$$

It is easy to see that to this order in η $H_{\rm eff}$ is Hermitian, however, to order η^2 we find

$$H_{\rm eff} = H_0 + \langle H'\mathcal{U}\rangle - \langle \mathcal{U}\left([H_0,\mathcal{U}] + H'\mathcal{U}\right)\rangle\,, \tag{16}$$

which is not Hermitian:

$$\begin{aligned} H_{\rm eff} - {H_{\rm eff}}^\dagger &= \left\langle H'\mathcal{U} + \mathcal{U}H' + \left[\mathcal{U}^2, H_0\right]\right\rangle + \cdots. \\ &= i\partial_t\left\langle \mathcal{U}^2\right\rangle + \left[\left\langle \mathcal{U}^2\right\rangle, H_0\right] + \cdots \end{aligned} \tag{17}$$

and equals, to this order the *total* time derivative of the operator $\left\langle \mathcal{U}^2\right\rangle$. This property corresponds to the fact that there is some probability "leakage" of order η^2 from the slowly varying part of the wave function to the rapidly varying ripple. This is to be expected since $\left\langle|\chi|^2\right\rangle = O(\eta^2)$ and is non-zero in general.

The non-Hermiticity of the effective Hamiltonian for the slowly varying modes frequently appears in expansions similar to the one considered here [5] [1]. This result can be better understood by considering the behavior of the above expansion under unitary transformations to which we now turn.

III. UNITARY TRANSFORMATIONS

In this section we determine the behavior of the effective Hamiltonian (16) under unitary transformations. We show below that the non-Hermitian piece in (16) is modified under such transformations and, in fact, can be completely eliminated.

For the case of a constant transformation, $\Psi \to \mathcal{C}\Psi$ with $\dot{\mathcal{C}} = 0$ it is clear that $H \to \mathcal{C}^\dagger H\mathcal{C}$ and $H_{\rm eff} \to \mathcal{C}^\dagger H_{\rm eff}\mathcal{C}$. If the unitary transformation is time-dependent, however, the result is more complicated. We will concentrate on transformations of the form

$$\Psi = e^F\hat{\Psi}, \tag{18}$$

where F is anti-Hermitian, rapidly varying, and of order η. The Hamiltonian for the transformed states $\hat{\Psi}$ is

$$\begin{aligned} \hat{H} &= e^{-F}He^F - ie^{-F}\partial_t e^F \\ &= H + [H,F] + \frac{1}{2}[[H,F],F] - i\dot{F} - \frac{i}{2}\left[\dot{F},F\right] - \frac{i}{6}\left[\left[\dot{F},F\right],F\right] + O(\eta^3). \end{aligned} \tag{19}$$

A tedious repetition of the procedure outlined in section II gives the following expression for the corresponding effective Hamiltonian

$$\begin{aligned} \hat{H}_{\rm eff} = H_0 &+ \langle H'\mathcal{U}\rangle + \frac{i}{2}\partial_t\left\langle F(F - 2\mathcal{U})\right\rangle + \frac{1}{2}\left\langle [H_0,[F,\mathcal{U}]]\right\rangle \\ &+ \frac{1}{2}\left\langle [[H_0,\mathcal{U}],\mathcal{U}]\right\rangle - \langle \mathcal{U}H'\mathcal{U}\rangle - \frac{1}{2}\left\langle \left[H_0,(F-\mathcal{U})^2\right]\right\rangle + O(H_0\eta^3, \Omega\eta^4). \end{aligned} \tag{20}$$

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[1] In [5] a non-Hermitian term was found already in the first order, the discrepancy between this result and the one obtained here is due to different assumptions concerning the time-dependence and magnitude of the various terms in the Hamiltonian, leading to different expansion parameters.

For a general choice of F this expression is still non-Hermitian. However for the special case

$$F = \mathcal{U} + O(\eta^2), \tag{21}$$

we obtain

$$\hat{H}_{\text{eff}} = H_0 + \frac{1}{2}\langle [H',\mathcal{U}]\rangle + \frac{1}{2}\langle [[H_0,\mathcal{U}],\mathcal{U}]\rangle - \langle \mathcal{U}H'\mathcal{U}\rangle + O(H_0\eta^3, \Omega\eta^4), \tag{22}$$

which is explicitly Hermitian and, in fact, it is identical to the Hermitian part of (16). It is, of course, always possible to return to the original frame using $\hat{\Psi} = \exp(-\mathcal{U} + \cdots)\Psi$.

The wave function in the new frame, $\hat{\Psi}$, has an expansion similar to (7)

$$\hat{\Psi} = \hat{\psi} + \hat{\chi}_1 + \hat{\chi}_2 + \cdots, \quad \hat{\chi}_n \sim \eta^n, \tag{23}$$

where the slowly varying piece $\hat{\psi}$ evolves unitarily in time since (22) is Hermitian (at least to order η^2). Using (11) we find

$$\begin{aligned} \hat{\psi} &= \left(1 - \frac{1}{2}\langle \mathcal{U}^2\rangle\right)\psi + O(\eta^3), \\ \hat{\chi}_1 &= 0. \end{aligned} \tag{24}$$

The $O(\eta^2)$ difference between ψ and $\hat{\psi}$ quantifies the probability leak into the rapidly varying sector for the original frame. The second order term in the wave function $\hat{\chi}_2$ cannot be determined without a specific choice for the $O(\eta^2)$ terms in F. The specific form of the relation between $\hat{\psi}$ and ψ can also be understood form the expression for the non-Hermitian part of the effective Hamiltonian obtained in (17).

We conjecture that this procedure can be carried order-by-order in η, but since we will not need these higher order corrections we will not pursue this further. The expression (22) is the form of the effective Hamiltonian which will be used in the following examples.

IV. EXAMPLES

The previous results can be applied to a variety of systems. In this section we will consider 5 such examples. Our main concern will be to illustrate a wide range of systems that can be studied using the above formalism

A. N-level quantum systems

In this section we consider a quantum system with a finite number of states. This serves, for example, as a model for spin or flavor changes in elementary particles; it also describes the basic physics of nuclear magnetic resonance and related phenomena. The most general Hamiltonian for these systems can be expanded in terms of the generators of $SU(N)$ which we denote by $\{\lambda^a\}$ and which satisfy

$$\left[\lambda^a, \lambda^b\right] = iC_{abc}\lambda^c. \tag{25}$$

We can then write

$$H_0 = \sum_a f^a\lambda^a, \qquad H_\omega = \sum_a g^a_\omega, \lambda^a \tag{26}$$

where H_ω are the Fourier coefficients of the rapidly-varying Hamiltonian (see (3)). Note that ${g^a_{-\omega}}^* = g^a_\omega$ since H' is Hermitian.

Substituting in (22) we find

$$\begin{aligned} H_{\text{eff}} &= \sum_a \varphi^a\lambda_a, \\ \varphi^a &= f^a - i\sum_{|\omega|>\Omega}\frac{{g^c_\omega}^* g^b_\omega}{2\omega}C_{abc} + \sum_{|\omega|>\Omega}\frac{f^d {g^c_\omega}^* g^b_\omega}{2\omega^2}C_{dbe}C_{eca} + \sum_{|\omega|,|\omega'|>\Omega}\frac{g^b_\omega g^c_{\omega'} {g^d_{\omega+\omega'}}^*}{3\omega'(\omega+\omega')}C_{dbe}C_{eca} + \cdots, \end{aligned} \tag{27}$$

where the ellipsis denote higher order terms in η.

For the particularly simple case of a two-level system with $C_{abc} = 2\epsilon_{abc}$ and $\lambda_a = \sigma_a$ (the usual Pauli matrices) this reduces to

$$H_{\rm eff} = \left(\mathbf{f} - \sum_{|\omega|>\Omega} \frac{\mathbf{g}_\omega \times \mathbf{g}_\omega^*}{\omega} + \cdots \right) \cdot \boldsymbol{\sigma}. \tag{28}$$

It is clear that there is the possibility for the H'-induced terms to generate resonances if we allow the f^a to vary slowly in time. To see this explicitly we consider (28) taking for simplicity $f^2 = g^3_\omega = 0$, and f_1 =constant. Substituting we find

$$\varphi_1 = f^1, \qquad \varphi_2 = 0, \qquad \varphi_3 = f^3 - E' + O(1/\Omega^2), \tag{29}$$

where

$$E' = 2 \sum_{|\omega|>\Omega} \frac{1}{\omega} \mathbf{Im} g^{2\,*}_\omega g^1_\omega. \tag{30}$$

If f^3 is allowed to vary slowly with time the effective Hamiltonian will exhibit a resonance when $f^3 = E'$ if $|E'| \gg |f^2|$. This resonance will lead to large transition amplitude provided f^3 varies sufficiently slowly: $|f^2| \gg \sqrt{|\dot{f}^3|}$ (which is the usual adiabatic resonance condition [6]). A specific example is presented in fig. 1.

In fig. 1 we also compare the results obtained using the effective Hamiltonian (28) to those obtained solving the Schrödinger equation exactly with initial conditions $\Psi(t=0) = \psi(t=0) = 1$. It can be seen that the solutions obtained using $H_{\rm eff}$ do indeed describe the average behavior of the wave-function provided η is sufficiently small. We conclude that, at least for the case of a 2-level system, the effective Hamiltonian (22) accurately describes the average evolution of the system.

The presence of noise (*i.e.*, H') can then generate unexpected resonances. The condition for these to occur is, qualitatively,

$$\Omega H_0 \sim H'^2; \tag{31}$$

we will see later that these resonances are related, but not identical, to the well-studied stochastic resonances [2].

B. δ-function comb

A simple model where the above formalism can be applied and which is also exactly solvable is provided by a 2-level system with Hamiltonian

$$H = \sum_n \sum_{i=1}^N \Lambda_i \delta(t - t_i - nT); \quad 0 < t_i < T, \tag{32}$$

where the matrices Λ_i are Hermitian. This represents a set of N δ functions which repeats with period T. This type of potential is of interest in signal processing [7] and is also similar to the one used in the study the effect of a laser beam on a set of charged particles [8].

For simplicity we will assume

$$\Lambda_i = \begin{pmatrix} 0 & \lambda_i \\ \lambda_i^* & 0 \end{pmatrix}; \qquad \sum_{i=1}^N \lambda_i = 0, \tag{33}$$

in this case $H_0 = 0$, so that $H = H'$, we will also take $\Omega = 2\pi/T$.

In order to obtain the effective Hamiltonian we first construct

$$-i \int_0^t dt\, H' = -i \sum_{i=1}^N \Lambda_i \Theta(t - t_i), \quad (0 < t < T), \tag{34}$$

where Θ denotes the step function. $\mathcal{U}$ is then the fast part of this quantity,

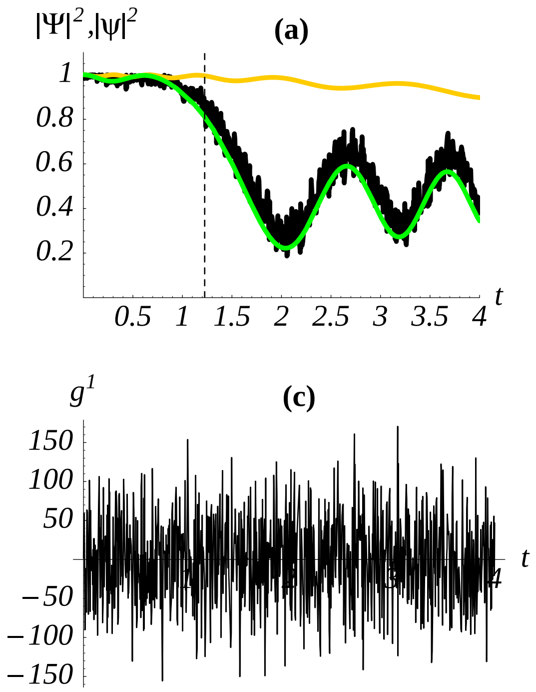

FIG. 1. Example of a resonant phenomenon induced by the presence of rapidly varying interactions of large amplitude. **(a):** comparison of $|\Psi|^2$ obtained by an exact (numerical) integration of Schrödinger's equation (black jagged curve) with $|\psi|^2$ obtained integrating (14) using the effective Hamiltonian in (28, 29) (light superimposed curve). The top light curve is the result of integrating Schrödinger's equation when $H' = 0$; the dotted vertical line denotes the time at which the diagonal elements in $H_{\rm eff}$ vanish. **(b):** Diagonal element of $H_{\rm eff}$; for this example we chose $E' = 4.78684$ and $f^3 = E'[2/(1+t^2) + 1/5]$. **(c,d):** Non vanishing elements of H'; the specific expression used was $g^1 + ig^2 = 39.9567e^{926.291t - 0.956304i} + 35.6145e^{984.461t + 0.660091i} + 39.1024e^{1057.84t - 0.732253i} + 30.4239e^{1208.99t + 2.43462i} + 29.1863e^{1953.06t + 0.719083i}$. For this example, $\eta \simeq 0.03$

$$\mathcal{U} = -i\int_0^t H' + i\left\langle \int_0^t H' \right\rangle = -i\sum_{i=1}^N \Lambda_i \left[\Theta(t-t_i) - 1 + \frac{t_i}{T}\right], \quad (0 < t < T), \tag{35}$$

where the slow part of a quantity is obtained by averaging over the period T.

Using then (33) and substituting in (22) we easily find

$$\begin{aligned} H_{\rm eff} &= \frac{1}{2}\left\langle [H', \mathcal{U}] \right\rangle + \cdots \\ &= \frac{1}{T}\left[\sum_{i>j} \mathbf{Im}\lambda_i\lambda_j^*\right]\sigma_3 + \cdots . \end{aligned} \tag{36}$$

For this system we have $\Omega \sim 1/T$ so that $\eta = \max\{|\lambda_i|\}$; it then follows that (36) will be accurate provided $|\lambda_i| \ll 1$.

This model can be solved exactly by elementary means. Replacing $\delta(t - t_i - nT)$ by a rectangle of height $1/\tau$ and width τ centered at $t = t_i + nT$ it is easy to see that in the limit $\tau \to 0$

$$\Psi(t_i^+) = e^{-i\Lambda_i}\Psi(t_i^-), \tag{37}$$

where $t_i^\pm = t_i \pm \delta,\ \delta \to 0$. It follows that

$$\Psi(T) = e^{-i\Lambda_N} e^{-i\Lambda_{N-1}} \cdots e^{-i\Lambda_1}\Psi(0). \tag{38}$$

In the limit where the λ_i are small we obtain

$$\begin{aligned} e^{-i\Lambda_N} e^{-i\Lambda_{N-1}} \cdots e^{-i\Lambda_1} &= \exp\left[-\frac{1}{2}\sum_{i>j}[\Lambda_i, \Lambda_j] + \cdots\right] \\ &= \exp\left[-iTH_{\rm eff} + \cdots\right] \end{aligned} \tag{39}$$

which shows that, at least for small λ_i, $H_{\rm eff}$ determines the leading contributions to the average time evolution of the wave function.

C. Geometrical Optics example

The calculations in the previous sections referred to quantum systems, but it clear that any system whose dynamical equations can be cast into a Schrödinger-like form can be treated in the same way. In particular for this general case there is no need to require H to be a Hermitian operator.

An example of this situation is provided by the description of light-ray evolution within geometrical optics [9]. For small angles the position and direction of light ray within geometrical optics can be described using a two-component vector

$$\begin{pmatrix} h \\ \alpha \end{pmatrix}, \tag{40}$$

where h denotes the height with respect to a reference line and α the tilt (assumed to be small). Any transformation of a light ray can be described using a 2×2 matrix [9]. In particular

$$\begin{aligned} \text{translation} : &\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}, \\ \text{refraction} : &\begin{pmatrix} 1 & 0 \\ (n_1/n_2 - 1)/R & n_1/n_2 \end{pmatrix}, \end{aligned} \tag{41}$$

where the translation is by a distance x and the refraction is from a medium of refraction index n_1 to another with index n_2 and R denotes the radius curvature of the interface.

We now assume that R and the index of refraction change smoothly though rapidly with distance. We define

$$\nu = -\frac{1}{n}\frac{dn}{dx}, \tag{42}$$

so that the general matrix which transports a ray by a distance δx is

$$M = \begin{pmatrix} 0 & 1 \\ \zeta' & \nu \end{pmatrix}. \tag{43}$$

where primes denote x derivatives and

$$\zeta = \int dx \frac{\nu}{R} \tag{44}$$

The system then corresponds to a two-level quantum system with "time" x and "Hamiltonian" $H = iM$. A simple application of (15) yields

$$H_{\rm eff} = i \begin{pmatrix} 0 & 1 \\ K^2 & 0 \end{pmatrix}, \qquad K^2 = \langle \zeta'(\ln n) \rangle = +\frac{1}{2} \left\langle (\ln n)^2 (1/R)' \right\rangle, \tag{45}$$

where we assumed $\langle \zeta' \rangle = \langle \nu \rangle = 0$ and we kept only the first corrections induced by the rapidly varying terms [2]. Note that K^2 can be negative and that it vanishes (at least to lowest order) when R or n are constant.

The effective operator which determines the translation over a finite distance X is then (assuming for simplicity that K is position independent)

$$A = e^{-iXH_{\rm eff}} = \begin{pmatrix} \cosh(KX) & (1/K)\sinh(KX) \\ K\sinh(KX) & \cosh(KX) \end{pmatrix} \tag{46}$$

(the $K^2 < 0$ case is obtained by analytic continuation). This matrix is equivalent to a thick symmetric lens with radius of curvature $\bar{R}$ and thickness $\bar{d}$ such that

$$\frac{1}{\bar{R}} = \frac{K}{1-\bar{n}} \tanh(KX/2) \qquad \bar{d} = \frac{\bar{n}}{K} \sinh(KX), \tag{47}$$

where $\bar{n}$ denotes the index of the lens material.

Thus, within the approximations inherent to geometrical optics, the effects of a region of rapidly varying index of refraction and curvature on light rays are equivalent, *on average* to those of a thick lens of appropriately chosen characteristics. For example, using a thick lens with a high index of refraction, $\bar{n} \gg 1$ we find

$$K|K| = -\frac{2\bar{n}^2}{\bar{R}\bar{d}}. \tag{48}$$

Conversely, a thick lens can be found which completely cancels the effects of A and this can be used to measure the fluctuations in the original system (more specifically, those fluctuations which contribute to ζ).

The above expressions suffer corrections form higher-order terms in the expansion in powers of η. Using (16) [3] we find that the next-order term in (45) is

$$i \begin{pmatrix} 0 & 0 \\ k^2 & 0 \end{pmatrix}, \quad k^2 = \left\langle \zeta^2 \right\rangle - \langle \zeta \rangle^2 - \frac{1}{2} \left\langle \ln n \left[\ln n - \frac{1}{2} \langle \ln n \rangle \right] \zeta' \right\rangle, \tag{49}$$

where we assumed for simplicity that $\langle (\ln n - \langle \ln n \rangle)\zeta \rangle$ and $\left\langle [\ln n - \langle \ln n \rangle]^2 \right\rangle$ are independent of x (in general the averaged quantities may still vary slowly with x). The second order correction is negligible provided $|K^2| \gg |k^2|$.

In addition there are deviations form these predications due to the inherent limitations of geometrical optics (for example, it is assumed the light rays lie on a plane, diffraction is neglected, etc.). In neglecting them we have tacitly assumed that the scale of all fluctuations is large compared to the wavelength and that all reflection and refraction angles are small.

[2] The second expression for K^2 follows from $2\nu(\ln n)/R = (1/R)'(\ln n)^2 - [(\ln n)^2/R]'$ and $\left\langle [(\ln n)^2/R]' \right\rangle = 0$ to first order in the rapidly varying quantities.

[3] As mentioned previously there is no reason to demand Hermiticity in the effective Hamiltonian for this case

D. The noisy Jaynes-Cummings model

In this example we consider a simplified version of an atom interacting with a photon field, as described by the Jaynes-Cummings model [10], with the addition of two types of interaction with an external rapidly varying fields. We will show that this problem is also well suited for study using the techniques introduced above. We first assume that the external field are coupled to the photons, and then directly to the atoms. We then show that both situations are unitarily equivalent.

The unperturbed Hamiltonian for this model is

$$H_0 = \omega_0 a^\dagger a + \frac{1}{2}\Omega_0(a^\dagger \sigma_- + a\sigma_+) + \epsilon\sigma_3. \tag{50}$$

Denoting by $|n;\uparrow\rangle$ and $|n;\downarrow\rangle$ the states with n photons and atomic spin up and down respectively, the eigenstates of H_0 are [10]

$$\begin{aligned} |n\pm\rangle &= c_{n\mp}|n;\downarrow\rangle \pm \mathrm{sign}(\kappa) c_{n\pm}|n-1;\uparrow\rangle, \\ c_{n\pm} &= \frac{1}{\sqrt{2}}\left(1 \pm \frac{1}{\sqrt{1+n\kappa^2}}\right); \qquad \kappa = \frac{\Omega_0}{2\epsilon - \omega_0}, \end{aligned} \tag{51}$$

with energies

$$E_{n\pm} = \left(n - \frac{1}{2}\right)\omega_0 \pm \left|\epsilon - \frac{1}{2}\omega_0\right| \sqrt{1+n\kappa^2}. \tag{52}$$

a. Noisy photon field We will now couple the photons to external sources which vary rapidly with time. The interaction Hamiltonian is assumed to be

$$H' = \xi a^\dagger + \xi^* a. \tag{53}$$

Substituting (50) and (53) in (22) yields

$$\hat{H_{\rm eff}} = H_0 + \omega_0 \left\langle |\theta|^2 \right\rangle - \mathbf{Im}\left\langle \theta^* \dot\theta \right\rangle + O(\eta^3); \qquad \xi = i\dot\theta. \tag{54}$$

The difference between $\hat{H_{\rm eff}}$ and H_0 is, in this case, trivial and can be eliminated by a simple change in the overall phase of the states.

Non-trivial terms may arise at higher orders but this would require calculating the (Hermitian version of the) effective Hamiltonian to order η^3. Instead of following this uninspiring approach we consider a different way of introducing the interaction with the external fields and then show that the corresponding effective operator corresponds to the order η^3 contribution generated by (53). To this end we first consider a unitary transformation of the form

$$S = \exp\left[a^\dagger\zeta(t) - a\zeta^*(t)\right]; \qquad i\dot\zeta - \omega_0\zeta + \xi = 0. \tag{55}$$

Then the transformed Hamiltonian is

$$\begin{aligned} H_{\rm new} &= S(H_0 + H')S^\dagger + i\dot S S^\dagger \\ &= H_0 - \frac{1}{2}\Omega_0\left(\sigma_-\zeta^* + \sigma_+\zeta\right) - \frac{1}{2}\mathbf{Im}(\zeta^*\dot\zeta) - \omega_0|\zeta|^2 \\ &= H_0 + H'_{\rm new}, \end{aligned} \tag{56}$$

which defines $H'_{\rm new}$. This shows that the original system is equivalent to one where the external sources are coupled directly to the spin of the atoms through ζ. From its definition it can be seen that ζ is of order η so that, substituting $H'_{\rm new}$ in (15), gives the effective Hamiltonian up to and including terms of order $\eta\zeta^2 \sim \eta^3$. The effects of this type of interaction are considered in the next paragraph.

b. Noisy spin interaction We now consider

$$H' = \zeta\sigma_+ + \zeta^*\sigma_-, \tag{57}$$

which to lowest order (using (15)) gives

$$H_{\rm eff} = H_0 - \mathbf{Im}\left\langle \vartheta^*\dot\vartheta \right\rangle \sigma_3; \qquad \zeta = i\dot\vartheta, \tag{58}$$

corresponding to the non-trivial replacement

$$\epsilon \to \epsilon_{\text{eff}} = \epsilon - \mathbf{Im}\left\langle \vartheta^* \dot{\vartheta} \right\rangle . \tag{59}$$

This replacement also describes the leading (average) effect of (53) provided we identify $\xi = \dddot{\vartheta}[1 + O(\eta)]$. If $\xi \sim \eta^0$ then the modification is indeed of order η^3. In order to go back to the original problem we act with S on the states.

The effects of the external sources is, to lowest order, summarized by the simple shift (59) which corresponds to a change in the energy gap between the two spin states of the "atom" of this model. In particular, for noise of sufficiently large amplitude, we can have $\epsilon_{\text{eff}} = \omega_0/2$ in which case the Rabi frequency vanishes, $E_{n+} = E_{n-}$ and photon number is conserved.

The shift in ϵ can also lead to resonant behavior between states of different n. For example the energies for the states $|0-\rangle$ and $|1-\rangle$ are equal provided

$$\epsilon_{\text{eff}} - \frac{1}{2}\omega_0 = \frac{\Omega_0^2}{8\omega_0} \pm \frac{1}{2}\omega_0 \tag{60}$$

which has a solution only for $0 < \omega_0/\Omega_0 \le 1/2$.

These results are reliable provided $\eta \ll 1$ which corresponds to $|\vartheta| \ll 1$ and $|\omega_0|,\ |\Omega_0|,\ |\epsilon| \ll \Omega$. As in the two level system resonances occur when $|H_0|/\Omega \sim \eta^2$.

E. Quantum system in an inhomogeneous potential

The ideas presented in the previous sections can be translated to the case of a particle whose Hamiltonian is of the form

$$H = -\frac{1}{2m}\nabla^2 + V_0 + V_1, \tag{61}$$

where V_1 varies rapidly with position. For this case we consider the time-independent Schrödinger equation $H\Psi = E\Psi$ and look for solutions $\Psi = \psi + \chi$ where χ is of small amplitude but exhibits rapid variation in position while ψ is slowly varying and of large amplitude. Substituting this Ansatz we find

$$\begin{aligned} \left(-\frac{1}{2m}\nabla^2 + V_0\right)\psi &\simeq E\psi - \langle V_1\chi\rangle , \\ -\frac{1}{2m}\nabla^2\chi &\simeq -V_1\psi \end{aligned} \tag{62}$$

which can be solved to lowest order giving

$$\begin{aligned} H_{\text{eff}}\psi &= E\psi, \\ H_{\text{eff}} &= -\frac{1}{2m}\nabla^2 + V_0 + \left\langle V_1 \frac{2m}{\nabla^2} V_1 \right\rangle . \end{aligned} \tag{63}$$

In this case, for any quantity A, $\langle A\rangle$ denotes the part of A (if any) which varies slowly with position.

In one dimension the same result can be obtained by converting the time-independent Schrödinger's equation to a first order equation for the vector $(\Psi, -id\Psi/dx)$ and substituting in (22) or (16), and using x as the evolution parameter.

The additional term in H_{eff} is *negative* definite and will tend to bind the particle. In particular, taking $V_0 = 0$ and assuming V_1 vanishes at infinity the effective Hamiltonian H_{eff} will always exhibit a bound state (of zero angular momentum) in ≤ 2 dimensions, that is, in ≤ 2 dimensions a rapidly varying potential of zero average will always exhibit localized states. The same is true in higher dimensions provided the amplitude of V_1 is large enough

V. PROBABILISTIC CONSIDERATIONS

In this section we provide an alternative view of the problem using the Fokker-Plank equation. For simplicity we consider the case of a two-level system with Hamiltonian

$$
\begin{aligned}
H &= H_0 + H', \\
H_0 &= \sum_a f^a \sigma_a, \\
H' &= \sum_a G^a \sigma_a,
\end{aligned}
\tag{64}
$$

where $\{G^a\}$ are stochastic variables whose probability function will be described below.

We will study the Fokker-Plank for the polarization vector $\psi^\dagger \boldsymbol{\sigma} \psi$ whose probability density is given by

$$
\mathcal{P}(\mathbf{r}, t) = \left\langle \delta^{(3)} \left(\psi^\dagger(t) \boldsymbol{\sigma} \psi(t) - \mathbf{r} \right) \right\rangle_G,
\tag{65}
$$

where the symbol $\langle \cdots \rangle_G$ denotes the average over the stochastic variables G^a. In terms of a functional integral we will use

$$
\langle A \rangle_G = \int \prod_a [dG^a] \; A \; \exp\left\{ -\frac{1}{2} \int\!\!\int dt \; dt' \sum_{ab} G^a(t) \mathcal{K}_{ab}(t, t') G^b(t') \right\},
\tag{66}
$$

with $\mathcal{K}$ symmetric ($\mathcal{K}_{ab}(t, t') = \mathcal{K}_{ba}(t', t)$) and positive definite. We denote by σ the inverse kernel $\mathcal{K}^{-1}$,

$$
\int ds \sum_c \mathcal{K}_{ac}(t, s) \sigma^{cb}(s, t') = \delta_a^b \delta(t - t').
\tag{67}
$$

It is easy to see that $\sigma^{ab}(t, t') = \left\langle G^a(t) G^b(t') \right\rangle$.

We will now restrict further considerations to cases where $G^a(t)$ is correlated with $G^b(t')$ only for t close to t', that is for cases where $\sigma_{ab}(t, t')$ vanishes except when $t \sim t'$. In this case we define

$$
\bar{\sigma}_{ab}(t) = \int_{-\infty}^{t} dt' \; \sigma_{ab}(t, t'),
\tag{68}
$$

and, following the standard derivation of the Fokker-Plank equation [11,12], we obtain,

$$
i\dot{\mathcal{P}} = \left(2 \sum_a L_a f^a - 4i \sum_{ab} L_a L_b \bar{\sigma}^{ab} \right) \mathcal{P},
\tag{69}
$$

where the f^a determine H_0 in (64) and $L_a, \; a = 1, 2, 3$ denote the usual angular momentum operators in 3 dimensions. There are corrections to this equation but these can be ignored provided $\sigma(t, t')$ is sufficiently localized around $t = t'$.

We will now restrict ourselves to situations where $\bar{\sigma}$ takes the form

$$
\bar{\sigma}^{ab} = \frac{1}{2} D \delta^{ab} - \frac{1}{2} \sum_c \epsilon^{abc} a_c.
\tag{70}
$$

The first term is commonly used in treating this type of problems, the second term implies a correlation between G^a and G^b with $a \neq b$ and is usually assumed to vanish; we will find, however that it is precisely this term that is responsible for the resonances described previously.

Substituting (70) in (69) yields

$$
i\dot{\mathcal{P}} = 2 \left[(\mathbf{f} - \mathbf{a}) \cdot \mathbf{L} - i D L^2 \right] \mathcal{P}.
\tag{71}
$$

It is important to note that this choice still corresponds to a positive definite kernel $\mathcal{K}$ so that (66) is well defined.

In order to relate these expressions to the ones obtained previously we first write $\bar{\sigma}$ in terms of the two-point correlator,

$$
\bar{\sigma}^{ab}(t, t') = \int_{-\infty}^{t} dt' \; \left\langle G^a(t) G^b(t') \right\rangle_G.
\tag{72}
$$

We then expand G^a in a Fourier series,

$$
G^a(t) = \sum_\omega G_\omega^a e^{-\imath \omega t}
\tag{73}
$$

(not necessarily restricted to $|\omega| > \Omega$) and assume that $\left\langle G^a_\omega G^b_{\omega'}\right\rangle_G \simeq 0$ for $\omega+\omega' \neq 0$ (which is equivalent to assuming that the correlator $\sigma^{ab}(t,t')$ is non-zero for $t \sim t'$ only). In this case

$$\begin{aligned} D &= \lim_{\delta\to 0} \frac{2}{3}\sum_\omega \frac{\delta}{\delta^2+\omega^2}\left\langle |\mathbf{G}_\omega|^2\right\rangle_G, \\ \mathbf{a} &= \lim_{\delta\to 0} i\sum_\omega \frac{\omega}{\delta^2+\omega^2}\langle \mathbf{G}_\omega \times \mathbf{G}_{-\omega}\rangle)_G. \end{aligned} \tag{74}$$

In obtaining these expressions we included a convergence factor $e^{\delta t}$, $\delta \to 0$ in (73). Note that D will vanish unless the G^a_ω are continuously distributed in an interval containing $\omega = 0$.

Comparing this result to (28) we find that the term $\mathbf{f} - \mathbf{a}$ in (71) corresponds to the leading term in $H_{\rm eff}$. In addition, however, the Fokker-Plank equation contains the non-Hermitian diffusion term $-2iDL^2\mathcal{P}$ which forces $\mathcal{P}$ to decrease exponentially in time for all but the zero-angular momentum modes [12]. Qualitatively this implies that for large times the polarization vector will end up uniformly distributed and the effects of the $\mathbf{f} - \mathbf{a}$ term will be completely washed-out. For intermediate times, however, the presence of the $\mathbf{f}-\mathbf{a}$ term can lead to interesting effects.

The general solutions to (71) can be obtained in terms of spherical harmonics [4] Y^l_m,

$$\mathcal{P} = \sum_{lm} \xi^l_m(t) e^{-2Dl(l+1)t - iv_0 mt} Y^l_m(\hat{\mathbf{r}}), \tag{75}$$

where the exponential is introduced for later convenience. Since $\mathcal{P}$ is real the coefficients ξ^l_m obey

$$\xi^{l\,*}_m = (-1)^m \xi^l_{-m}. \tag{76}$$

The above expansion can be substituted into (71) leading, for each l, to a set of $2l+1$ coupled ordinary differential equations in t that can in principle be solved for any choice of $\mathbf{f}$, $\mathbf{a}$ and D.

To illustrate this procedure we will consider a special case which is similar to the one often studied when considering stochastic resonances. We take

$$\mathbf{f} - \mathbf{a} = \frac{1}{2} v_0 \hat{\mathbf{z}} + \frac{1}{2} u\cos(\omega_0 t)\hat{\mathbf{x}}, \tag{77}$$

corresponding, for example, to $f^3 = G^3 = 0$ (so that $a^1 = a^2 = 0$), $f^2 = 0$, $f^1 = u\cos(\omega_0 t)$ and $a^3 = -v_0$. The equations for the coefficients ξ^l_m in (75) then become

$$\dot{\xi}^l_m + \frac{i}{2} u\cos(\omega_0 t)\left[\sqrt{l(l+1)-m(m-1)}e^{iv_0 t}\xi^l_{m-1} + \sqrt{l(l+1)-m(m+1)}e^{-iv_0 t}\xi^l_{m+1}\right] = 0, \tag{78}$$

For $l = 0$ the solution is simply ξ^0_0 =constant which is determines the normalization of $\mathcal{P}$; the equations for $l \neq 0$ can be solved numerically using standard techniques. The case $l = 1$ is of special interest since the coefficients $\xi^{l=1}_m$ determine the average polarization of the system as a function of time:

$$\begin{aligned} \left\langle \psi^\dagger(t)\boldsymbol{\sigma}\psi(t)\right\rangle &= \int d^2\hat{\mathbf{r}}\,(\mathcal{P}\hat{\mathbf{r}}) \Big/ \int d^2\hat{\mathbf{r}}\mathcal{P} \\ &= \frac{\sqrt{2/3}}{\xi^0_0} e^{-4Dt}\left(\mathbf{Re}\left[\xi^1_{-1}(t)e^{iv_0 t}\right], \mathbf{Im}\left[\xi^1_{-1}(t)e^{iv_0 t}\right], \frac{1}{\sqrt{2}}\xi^1_0(t)\right). \end{aligned} \tag{79}$$

We will be interested in the possibility of the system resonating at the driving frequency ω_0. This can be investigated by first solving the above equations and then Fourier transforming the result. We define,

$$\tilde{\xi}^l_m(\omega) = \int dt e^{i\omega t}\xi^l_m(t), \tag{80}$$

and we study the behavior of $\left|\xi^1_1(\omega = v_0)\right|$ as a function of v_0 for various values of ω_0 (note that $\tilde{\xi}$ also depends explicitly on v_0 since this parameter appears in the differential equation (78)). The result is presented in Fig. 2 which clearly shows an enhancement in the Fourier coefficients of frequency v_0 when $v_0 = \omega_0$, the shape of the curves are characteristic of resonant behavior. These resonances are also illustrated by the behavior of $\left\langle \psi^\dagger\sigma^3\psi\right\rangle$ for various values of v_0, ω_0. An example is plotted in fig. 3.

---

[4] It is easy to see that $\mathcal{P}$ is, in fact, independent of $|\mathbf{r}|$.

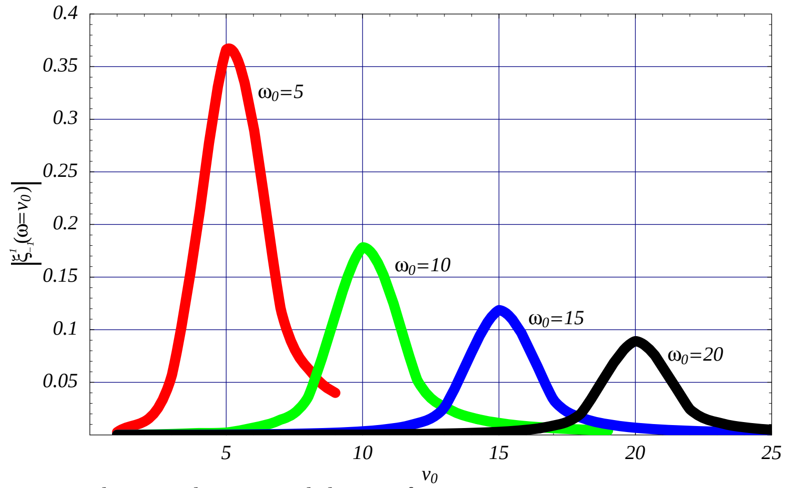

FIG. 2. Resonances in a two-level system; ω_0 denotes the external driving frequency.

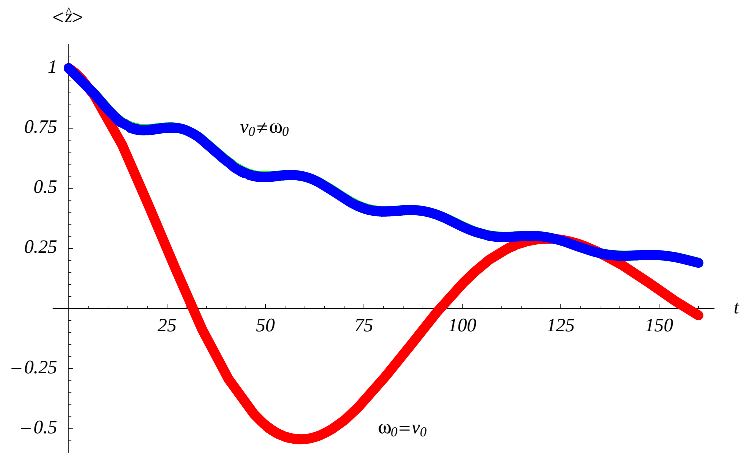

FIG. 3. Resonant behavior of the z component polarization vector (79), $\left\langle \psi^\dagger \sigma^3 \psi \right\rangle = \langle \hat{z} \rangle$, for a two level system. The parameters chosen were $u = 0.1$, $\xi_0^0 = 1/\sqrt{3}$, $D = 0.0025$, $\omega_0 = 5$ and, for the case $v_0 \neq \omega_0$, $v_0 = \omega_0 \pm 0.2$

The same system can be studied using the time averaging procedure of sections II-III provided we assume that the corresponding restrictions on the parameters are satisfied. Assuming this is the case, the effective Hamiltonian corresponding to the choice (77) is readily seen to be, to lowest order in η,

$$H_{\rm eff} = \begin{pmatrix} v_0/2 & \cos(\omega_0 t) \\ \cos(\omega_0 t) & -v_0/2 \end{pmatrix} . \tag{81}$$

The corresponding Schrödinger equation for the slow modes $H_{\rm eff}\psi = i\dot{\psi}$ can be solved numerically using standard techniques and the solutions are seen to exhibit resonances whenever v_0 is an integral multiple of ω_0.

There are, of course, differences between the solutions to the Schrödinger equation associated with (81) and the solutions derived from (78). The quantities G^a used in obtaining (78) are assumed to be stochastic variables whose distribution is determined by (66, 67, 68, 70). In contrast, when deriving (81) we assumed the g^a_ω are non-zero only for $|\omega| > \Omega$, and we also required $\Omega \gg |\omega_0|, \ |v_0|, \ |\dot{g}_\omega/g_\omega|$.

The values for **a** obtained in cases become identical if we assume that the average over the stochastic variables in the first case give the same result as the average over time intervals much larger than $1/\Omega$ in the second case.

The diffusion term D will vanish, as mentioned above, unless the G_ω are distributed continuously around $\omega = 0$. This term corresponds to the non-Hermitian contribution (17) which, for this case is simply proportional to the unit matrix. Taking $f^{2,3} = g^3 = 0$, in the example of section IV A (for the case $N = 2$) we find

$$H_{\rm eff} - {H_{\rm eff}}^\dagger = -id, \quad d = \frac{d}{dt} \sum_{|\omega|>\Omega} \left| \frac{g^1_\omega + i g^2_\omega}{\omega} \right|^2 , \tag{82}$$

which is of order η^2. Note that $d = 0$ if $g^{1,2}_\omega$ are time independent; this corresponds to the vanishing of D should G_ω vanish when ω lies in an interval around $\omega = 0$.

In concluding this section we note that it is possible to generate the required correlation between g^1 and g^2 by mixing and filtering two uncorrelated functions $n^{1,2}$. The details are presented in the Appendix.

VI. COMPARISON WITH THE STANDARD STOCHASTIC RESONANCES

The resonances described above are reminiscent of the well-studied stochastic resonances [2] that are characterized by an increased sensitivity to small perturbations when noise of an optimal amplitude is introduced. This feature is also observed when the condition (31) is satisfied, and is illustrated by the behavior of the system studied in the previous section.

More specifically, stochastic resonances occur when there is a match between a noise-induced transition rate, r_N and the one produced by an external perturbation. If the latter is assumed to be harmonic of frequency ω_0, then typically resonances occur when $\omega_0 \sim \pi r_N$. This can be understood by considering a system that initially has 2 degenerate minima, such that the harmonic perturbation will first favor one and then the other (alternating with period π/ω_0). If the resonance condition on r_N is realized, then the times at which one minima is disfavored will coincide with the times at which noise-induced transitions to the other minima are most probable, and this enhances the response of the system to the external perturbation. This behavior is also observed in the systems studied above, for example, the resonances in Fig. 2 occur when $v_0 = \omega_0$ where, according to (81), v_0 is proportional to the noise-induced transition rate.

There are, however, some technical differences. To illustrate these we consider the following one-dimensional system that exhibits stochastic resonances

$$\dot{x} = V'(x,t) + e(t), \quad V(x,t) = V_0(x) + u\, x\, \cos(\omega_0 t), \tag{83}$$

where e denotes a stochastic variable, u is a small coupling constant and V_0 is a potential with two (degenerate) minima. The noise is assumed to obey

$$\langle e(t)e(t')\rangle = F\delta(t-t'). \tag{84}$$

Following the same steps [13] described above it is possible to obtain the Fokker-Plank equation for the probability density $\mathcal{P}(y,t) = \langle \delta\,(x(t)-y)\rangle_e$ and the corresponding average $\langle x(t)\rangle_e$. The Fourier coefficient of $\langle x(t)\rangle_e$ corresponding to frequency ω_0 has an amplitude proportional to $(\lambda/F)/\sqrt{\omega_0^2+\lambda^2}$ where λ denotes the noise-induced hopping rate (the Kramers' rate [14]), $\ln\lambda \propto -1/F$. For fixed ω_0 this amplitude also displays an enhancement at a certain value of F [2]. Comparing these results with those obtained in the previous section we note that

- The usual stochastic resonances occur for uncorrelated noise obeying (84) while resonance behavior in (64) requires the correlations implied by having $\mathbf{a} \neq 0$ in (70).
- The resonances described above have the usual shape (see Fig. 2) for the resonant curve. This is not necessarily the case for the stochastic resonances usually discussed in the literature.
- Usual stochastic resonances occur whenever the driving frequency is about half the Kramers' rate, which depends exponentially on the noise level F. For the case presented in this paper resonances occur when the driving frequency is $\sim |\mathbf{a}|$ as defined in (74), and is proportional to the square of the amplitude of the stochastic variables G^a.

VII. CONCLUSIONS

In this work we propose a formalism which makes possible to study systems under the influence of rapidly-varying external fields (which are not necessarily perturbative) whose typical frequency we denoted by Ω. The formalism provides a solution as a power series in $1/\Omega$ and assumes a clear separation of fast (frequencies $\gtrsim \Omega$) and slow (frequencies $\ll \Omega$) modes.

We showed that the evolution of the slow modes is determined by an effective Hamiltonian which is not necessarily Hermitian; a point noted in other related calculations [5]. The non-Hermitian contributions to the effective Hamiltonian, however, can be eliminated by performing an appropriate unitary transformation.

The formalism was applied to various classical and quantum systems. In some examples we found that the external field can produce a resonant behavior in the system. These resonances are related, but not identical, to the stochastic resonances studied in the literature [2] In particular the resonant phenomena studied in this paper occur only when the interaction with the environment involves several correlated terms.

In one particular application of the formalism we argued that the presence of a random time-independent potential will necessarily generate bound states in systems of dimension 1 and 2, and in other dimensions as well provided the amplitude of the potential is sufficiently large. The connection of this result with the phenomenon of Anderson localization [15] are tantalizing and will be considered in a future publication.

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We would like to thank W. Beyermann, R. deCoss and T.J. Weiler for illuminating comments and insights. This research was supported in part by US DOE contract number DE-FG03-94ER40837(UCR) and by Conacyt (México).

APPENDIX

In this appendix we describe a simple construction which generates stochastic variables g^a satisfying (70,72), in terms of a set of uncorrelated variables n^i, specifically, we assume

$$\langle n_i(t) n_j(t') \rangle = \frac{1}{2} D_i \delta_{ij} \delta(t-t'), \tag{85}$$

and search for new variables g^a satisfying

$$\left\langle g^a(t) g^b(t') \right\rangle = \frac{1}{2} F \mathcal{E}(t-t') \delta_{ab} - \frac{1}{2} \epsilon_{abc} a^c \mathcal{O}(t-t'), \tag{86}$$

where $\mathcal{E}$ is an even function of its argument while $\mathcal{O}$ is odd and satisfy

$$\int_{-\infty}^{0} \mathcal{E}(s) ds = \int_{-\infty}^{0} \mathcal{O}(s) ds = 1. \tag{87}$$

In terms of the Fourier transformed quantities,

$$\tilde{g}^a(\omega) = \int_{-\infty}^{\infty} dt\, e^{-i\omega t} g^a(t), \quad \tilde{n}_i(\omega) = \int_{-\infty}^{\infty} dt\, e^{-i\omega t} n_i(t), \tag{88}$$

we require

$$\left\langle \tilde{g}^a(\omega)\tilde{g}^b(\omega')\right\rangle = \pi\delta(\omega+\omega')\left[F\delta_{ab}\tilde{\mathcal{E}}(\omega) - \epsilon_{abc}a^c\tilde{\mathcal{O}}(\omega)\right],$$
$$\langle \tilde{n}_i(\omega)\tilde{n}_j(\omega')\rangle = \pi\delta(\omega+\omega')D_i\delta_{ij}, \tag{89}$$

where

$$\tilde{\mathcal{E}}(\omega)^* = \tilde{\mathcal{E}}(-\omega) = +\tilde{\mathcal{E}}(\omega),$$
$$\tilde{\mathcal{O}}(\omega)^* = \tilde{\mathcal{O}}(-\omega) = -\tilde{\mathcal{O}}(\omega). \tag{90}$$

We look for a linear relation between $\tilde{g}^a$ and $\tilde{n}_i$, namely

$$\tilde{g}^a(\omega) = \sum_i K_{ai}(\omega)n_i(\omega). \tag{91}$$

Writing

$$K_{ia} = \frac{1}{\sqrt{D_aF\tilde{\mathcal{E}}}}\left(Q_+ + Q_-\right)_{ia}, \tag{92}$$

with $Q_\pm(\pm\omega) = \pm Q_\pm(\omega)$ and assuming $Q_\pm^T = \pm Q_\pm$ (where T indicates the transpose) we find

$$Q_+^2 + Q_-^2 = 1, \qquad \{Q_+, Q_-\}_{ij} = \epsilon_{ikj}\nu^k, \tag{93}$$

where $\nu^k = a^k\tilde{\mathcal{O}}/(F\tilde{\mathcal{E}})$. These equations are solved, for example, by choosing

$$Q_+ = \cosh u\left(1 + \hat{\mathbf{a}}\otimes\hat{\mathbf{a}}\right), \qquad (Q_-)_{ij} = \sinh u\epsilon_{ikj}\hat{a}^k \tag{94}$$

with $\sinh(2u) = |\nu|$.

It follows that given a set of uncorrelated variables n_i it is possible to generate the desired correlated quantities g^a through a linear filter defined by the (frequency-dependent) matrix K.

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